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1995 J. Phys. A: Math. Gen. 28 6405

(http://iopscience.iop.org/0305-4470/28/22/016)

J. Phys. A: Math. Gen. 28 (1995) 6405–6415. Printed in the UK

Space groups on the quantum torus

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Received 9 May 1995, in final form 9 August 1995

Abstract. Space groups in two dimension arise from the commutative translation group Z^2 and its automorphisms in $GL(2, Z)$. From the free group F_2 and its automorphisms we construct 17 non-commutative groups and their homomorphisms to the 17 space groups with commutative translations.

1. Introduction

Any crystallographic space group G in 2D acts on the plane R^2 and admits a translation group with two commuting and independent generators. The translation group is isomorphic to Z^2. The transversal (set of orbit representatives) of the action of the subgroup $Z^2 < G$ on R^2 is the unit cell R^2/Z^2 which, upon appropriate identification of the edges, becomes the *torus*. The two generators of Z^2 generate the homotopy group of the torus.

A non-commutative scheme is obtained if the translation group Z^2 is replaced by the free group F_2 with two generators $\langle x_1, x_2\rangle$. In the context of non-commutative geometry with operator algebras proposed by Connes [1], the action and group algebra of F_2 have led to the notion of a '*quantized torus*', cf Effros [2] and [1, p340–7]. Note that this differs from the concept of a quantum group which involves a deformation parameter.

In crystallographic terms, the group Z^2 associated with the torus is denoted by $P1$. We denote its non-commutative generalization associated with the quantum torus by $\mathcal{P}1 := F_2$ and look for other non-commutative space groups. We construct groups which

(1) admit an at least two-generator subgroup of F_2, and
(2) admit a homomorphism to one of the 17 space groups of 2D crystallography.

2. The automorphisms of the free group

We shall work with elements and subgroups of $\mathrm{Aut}(F_n)$ and its action on F_n. Nielsen [4] showed that $\mathrm{Aut}(F_n)$ is finitely generated. We shall use the involutive generators and the subgroup relations given in [5]. For general n, the group F_n is isomorphic to $\mathrm{Inn}(F_n)$. We denote by $\langle T_1, T_2, \ldots\rangle$ the images in $\mathrm{Aut}(F_n)$ of the generators $\langle x_1, x_2, \ldots\rangle$ under this isomorphism and interpret them as non-commutative translations. There are no finite-order elements in $\mathrm{Inn}(F_n)$, so we must look for them in the cosets. A method for finding the elements $g \in \mathrm{Aut}(F_n)$ of finite order is described by McCool [7]. For $H < \mathrm{Aut}(F_n)$ of finite order, we can construct the semidirect product subgroup $(\mathrm{Inn}(F_n) \times_s H) < \mathrm{Aut}(F_n)$ which we call a symmorphic NC space group. The relation to crystallographic space groups

0305-4470/95/226405+11$19.50

is governed by the two homomorphisms

$$\text{hom}_1: \quad F_n \to Z^n$$
$$\text{hom}_2: \quad \text{Aut}(F_n) \to GL(n, Z).$$

Given an element of F_n, that is a word $w(x_1, x_2, \ldots)$, its image under the Abelianization hom_1 is found from the power sums $(n_1(w), n_2(w), \ldots)$ of the generators $x_1, x_2, \ldots$ in w. An automorphism $\phi \in \text{Aut}(F_n)$ is specified by giving the images $y_1(x_1, x_2, \ldots), y_2(x_1, x_2, \ldots), \ldots$. We compose automorphisms according to Nielsen [4]. The image of ϕ under hom_2 is an $n \times n$ element from the matrix group $GL(n, Z)$ whose entries are the power sums. In what follows we consider $n = 2, 3$.

3. 17 non-commutative space groups

The homomorphism hom_2 allows one to search for preimages of the space groups in the plane. A candidate for a non-commutative space group found in this way will be denoted by $\mathcal{G} < \text{Aut}(F_2)$. The 17 space groups in the plane were described in terms of sets of generators and relations by Coxeter and Moser [8]. We construct a preimage $\mathcal{G}$ for each space group G, with the same set of generators as Coxeter and Moser. The relators $\mathcal{R}$ are checked within $\text{Aut}(F_2)$ and fall into two classes: relators $\mathcal{R}_1$ equal to unity both in $\mathcal{G}$ and in G, and relators $\mathcal{R}_2$, equal to unity in G. The pairs of space groups are then $\mathcal{G} = \langle \ldots | \mathcal{R}_1 \rangle$ and $G = \langle \ldots | \mathcal{R}_1, \mathcal{R}_2 \rangle$, respectively. The relators $\mathcal{R}_2$ determine in $\mathcal{G}$ the kernel $\ker(\text{hom}) \lhd \mathcal{G}$ of the specific homomorphism $\text{hom}: \mathcal{G} \to G$. We show that the relators $\mathcal{R}_2$ can always be reduced to the commutator $K((T_1)^{\pm 1}, (T_2)^{\pm 1})$, even if T_1, T_2 do not belong to the non-commutative translation group of $\mathcal{G}$. Any space group G is now a factor group

$$G = \mathcal{G}/\ker(\text{hom}).$$

We give the 17 space groups in the order and notation of the International tables [9]. For each group we first give alternative sets of generators $\langle a \rangle$, $\langle b \rangle, \ldots$, as in [8], with some changes of notation to avoid symbols used for $\text{Aut}(F_2)$. We use alternative sets of generators to display representative elements for each space group. Under $(\text{hom})^{-1}$ we give in the next row a preimage in $\text{Aut}(F_2)$ for each generator, compare also section 4. Under $\mathcal{R}$ we give the relators $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$. In the row that follows we give under ker for the relators $\mathcal{R}_2$ their expressions in $\text{Aut}(F_2)$. They determine elements of infinite order in $\mathcal{G}$.

$\mathcal{P}1$

$\langle a \rangle$	X	Y	
$(\text{hom})^{-1}$	T_1	T_2^{-1}	(1)
$\mathcal{R}$	$XYX^{-1}Y^{-1}$		
ker	$K(T_1, T_2^{-1})$		

$\mathcal{P}2$

$\langle a \rangle$	X	Y	T		
$(\text{hom})^{-1}$	T_1	T_2^{-1}	$\sigma_1\sigma_2$		
$\mathcal{R}$	T^2	$TXTX$	$TYTY$	$XYX^{-1}Y^{-1}$	
ker				$K(T_1, T_2^{-1})$	(2)
$\langle b \rangle$	$U_1 = YT$	$U_2 = T$	$U_3 = XU_2$		
$(\text{hom})^{-1}$	S_3	S_2	S_1		
$\mathcal{R}$	$(U_1)^2$	$(U_2)^2$	$(U_3)^2$	$(U_1U_2U_3)^2$	
ker				$K(T_2^{-1}, T_1)$	

$\mathcal{P}m$

$\langle a\rangle$	X	Y	R		
$(\mathrm{hom})^{-1}$	T_1	T_2^{-1}	σ_1		
$\mathcal{R}$	R^2	$RXRX$	$RYRY^{-1}$	$XYX^{-1}Y^{-1}$	
ker				$K(T_1, T_2^{-1})$	(3)
$\langle b\rangle$	R	$R' = RX$	Y		
$(\mathrm{hom})^{-1}$	σ_1	$\sigma_1 T_1$	T_2^{-1}		
$\mathcal{R}$	R^2	$(R')^2$	$RYRY^{-1}$	$R'YR'Y^{-1}$	
ker				$K(T_1^{-1}, T_2^{-1})$	

$\mathcal{P}g$

$\langle a\rangle$	X	P	
$(\mathrm{hom})^{-1}$	$T_2 T_1$	$T_1 c_{13}$	
$\mathcal{R}$	$P^{-1}XPX$		
ker	$K(T_2, T_1^{-1})$		(4)
$\langle b\rangle$	P	$Q = PX$	
$(\mathrm{hom})^{-1}$	$T_1 c_{13}$	$c_{13} T_1$	
$\mathcal{R}$	$P^2 Q^{-2}$		
ker	$K(T_1, T_2^{-1})$		

$\mathcal{C}m$

$\langle a\rangle$	P	Q	R	
$(\mathrm{hom})^{-1}$	$T_1 c_{13}$	$c_{13} T_1$	c_{13}	
$\mathcal{R}$	R^2	$RPRQ^{-1}$	$P^2 Q^{-2}$	
ker			$K(T_1, T_2^{-1})$	(5)
$\langle b\rangle$	R	$S = PR$		
$(\mathrm{hom})^{-1}$	c_{13}	T_1		
$\mathcal{R}$	R^2	$(SR)^2(RS)^{-2}$		
ker		$K(T_1, T_2^{-1})$		

$\mathcal{P}2mm$

$\langle a\rangle$	Y	R	R'	R_2	
$(\mathrm{hom})^{-1}$	T_1^{-1}	$\sigma_2 T_2$	σ_2	σ_1	
$\mathcal{R}$	R^2	$(R')^2$	$(R_2)^2$	$R'YR'Y^{-1}$	
ker					
$\mathcal{R}$	$R_2 R R_2 R$	$R_2 Y R_2 Y$	$RYRY^{-1}$	$R_2 R' R_2 R'$	
ker			$K(T_2^{-1}, T_1^{-1})$		(6)
$\langle b\rangle$	$R_1 = R$	R_2	$R_3 = R'$	$R_4 = R_2 Y$	
$(\mathrm{hom})^{-1}$	$\sigma_2 T_2$	σ_1	σ_2	$T_1 \sigma_1$	
$\mathcal{R}$	$(R_1)^2$	$(R_2)^2$	$(R_3)^2$	$(R_4)^2$	
ker					
$\mathcal{R}$	$(R_1 R_2)^2$	$(R_2 R_3)^2$	$(R_3 R_4)^2$	$(R_4 R_1)^2$	
ker				$K(T_1, T_2^{-1})$	

$\mathcal{P}2mg$

$\langle a \rangle$	P	Q	R		
$(\mathrm{hom})^{-1}$	$T_1 c_{13}$	$c_{13} T_1$	m''		
$\mathcal{R}$	R^2	$RPRP$	$RQRQ$	$P^2 Q^{-2}$	
ker				$K(T_1, T_2^{-1})$	(7)
$\langle b \rangle$	$V_1 = PR$	$V_2 = QR$	R		
$(\mathrm{hom})^{-1}$	$T_1 c_{13} m''$	$c_{13} T_1 m''$	m''		
$\mathcal{R}$	$(V_1)^2$	$(V_2)^2$	R^2	$(V_2 R V_2)^{-1}(V_1 R V_1)$	
ker				$K(T_2^{-1}, T_1)$	

$\mathcal{P}2gg$

$\langle a \rangle$	P	Q	T	
$(\mathrm{hom})^{-1}$	$T_1 c_{13}$	$c_{13} T_1$	$c_{13} m''$	
$\mathcal{R}$	T^2	$TPTQ$	$P^2 Q^{-2}$	
ker			$K(T_1, T_2^{-1})$	(8)
$\langle b \rangle$	P	$O = PT$		
$(\mathrm{hom})^{-1}$	$T_1 c_{13}$	$T_1 m''$		
$\mathcal{R}$	$(PO)^2$	$(P^{-1} O)^2$		
ker	$K(T_1, T_2^{-1})$			

$C2mm$

$\langle a \rangle$	R_1	R_2	R_3	R_4	T		
$(\mathrm{hom})^{-1}$	$S_1 m'' S_1$	c_{13}	m''	$S_1 c_{13} S_1$	S_1		
$\mathcal{R}$	$(R_1)^2$	$(R_2)^2$	$(R_3)^2$	$(R_4)^2$	T^2		
ker							
$\mathcal{R}$	$(R_1 R_2)^2$	$(R_2 R_3)^2$	$(R_3 R_4)^2$	$(R_4 R_1)^2$	$T R_1 T R_3$	$T R_2 T R_4$	(9)
ker	$K(T_1, T_2^{-1})$		$K(T_2, T_1)$				

$\langle b \rangle$	R_1	R_2	T		
$(\mathrm{hom})^{-1}$	$S_1 m'' S_1$	c_{13}	S_1		
$\mathcal{R}$	$(R_1)^2$	$(R_2)^2$	T^2	$(R_1 R_2)^2$	$(R_1 T R_2 T)^2$
ker				$K(T_1, T_2^{-1})$	

$\mathcal{P}4$

$\langle a \rangle$	U_1	U_2	U_3	U_4	S	
$(\mathrm{hom})^{-1}$	S_3	$S_2 S_1 S_2$	$S_2 S_3 S_2$	S_1	$c_{13} \sigma_1$	
$\mathcal{R}$	$(U_1)^2$	$(U_2)^2$	$(U_3)^2$	$(U_4)^2$		
ker						
$\mathcal{R}$	S^4	$S^{-i} U_4 S^i U_i^{-1}$	$U_1 U_2 U_3 U_4$			(10)
ker			$K(T_2^{-1}, T_1)$			

$\langle b \rangle$	S	$T U_4$	
$(\mathrm{hom})^{-1}$	$c_{13} \sigma_1$	S_1	
$\mathcal{R}$	S^4	T^2	$(ST)^4$
ker			$K(T_2, T_1)$

$\mathcal{P}4mm$

$\langle a\rangle$	R_1	R_2	R_3	R_4	R		
$(\text{hom})^{-1}$	$\sigma_2 T_2$	σ_1	σ_2	$T_1\sigma_1$	c_{13}		
$\mathcal{R}$	$(R_1)^2$	$(R_2)^2$	$(R_3)^2$	$(R_4)^2$	R^2		
ker							(11)
$\mathcal{R}$	$(R_1R_2)^2$	$(R_2R_3)^2$	$(R_3R_4)^2$	$(R_4R_1)^2$			
ker				$K(T_1, T_2^{-1})$			
$\mathcal{R}$	RR_1RR_4	RR_2RR_3					
ker							
$\langle b\rangle$	R	R_1	R_2				
$(\text{hom})^{-1}$	c_{13}	$\sigma_2 T_2$	σ_1				
$\mathcal{R}$	R^2	$(R_1)^2$	$(R_2)^2$	$(R_1R_2)^2$	$(R_2R)^4$	$(RR_1)^4$	
ker						$K(T_1, T_2^{-1})$	

$\mathcal{P}4gm$

$\langle a\rangle$	R_1	R_2	R_3	R_4	S	
$(\text{hom})^{-1}$	$\sigma_2 T_2$	$\sigma_1 T_1$	$T_2\sigma_2$	$T_1\sigma_1$	$c_{13}\sigma_1$	
$\mathcal{R}$	$(R_1)^2$	$(R_2)^2$	$(R_3)^2$	$(R_4)^2$	S^4	
ker						
$\mathcal{R}$	$S^{-i}R_4S^iR_i^{-1}$	$(R_1R_2)^2$	$(R_2R_3)^2$	$(R_3R_4)^2$	$(R_4R_1)^2$	(12)
ker		$K(T_2^{-1}, T_1^{-1})$	$K(T_1^{-1}, T_2)$	$K(T_2, T_1)$	$K(T_1, T_2^{-1})$	
$\langle b\rangle$	$R = R_4$	S				
$(\text{hom})^{-1}$	$T_1\sigma_1$	$c_{13}\sigma_1$				
$\mathcal{R}$	R^2	S^4	$(RS^{-1}RS)^2$			
ker			$K(T_1, T_2^{-1})$			

$\mathcal{P}3$

$\langle a\rangle$	U_1	U_2	U_3		
$(\text{hom})^{-1}$	$c_{12}c_{23}$	$c_{12}c_{23}T_1$	$T_1^{-1}c_{12}c_{23}$		
$\mathcal{R}$	$(U_1)^3$	$(U_2)^3$	$(U_3)^3$	$U_1U_2U_3$	
ker			$K(T_1^{-1}, T_2^{-1})$		(13)
$\langle b\rangle$	U_1	U_2			
$(\text{hom})^{-1}$	$c_{12}c_{23}$	$c_{12}c_{23}T_1$			
$\mathcal{R}$	$(U_1)^3$	$(U_2)^3$	$(U_1U_2)^3$		
ker			$K(T_2^{-1}, T_1^{-1})$		

$\mathcal{P}3m1$

$\langle a\rangle$	U_1	U_2	R				
$(\text{hom})^{-1}$	$c_{12}c_{23}$	$c_{12}c_{23}T_1$	c_{13}				
$\mathcal{R}$	$(U_1)^3$	$(U_2)^3$	$(U_1U_2)^3$	R^2	RU_1RU_1	RU_2RU_2	
ker			$K(T_2^{-1}, T_1^{-1})$				
$\langle b\rangle$	$R_1 = RU_2$	$R_2 = U_1R$	$R_3 = R$				(14)
$(\text{hom})^{-1}$	$c_{12}T_1$	c_{23}	c_{13}				
$\mathcal{R}$	$(R_1)^2$	$(R_2)^2$	$(R_3)^2$				
ker							
$\mathcal{R}$	$(R_1R_2)^3$	$(R_2R_3)^3$	$(R_3R_1)^3$				
ker	$K(T_1^{-1}, T_2^{-1})$						

$\mathcal{P}31m$

$\langle a\rangle$	U_1	U_2	R			
$(\mathrm{hom})^{-1}$	$c_{12}c_{23}$	$c_{12}c_{23}T_1$	m''			
$\mathcal{R}$	$(U_1)^3$	$(U_2)^3$	R^2	RU_1RU_2	$(U_1U_2)^3$	
ker					$K(T_2^{-1}, T_1^{-1})$	(15)
$\langle b\rangle$	U_1	R				
$(\mathrm{hom})^{-1}$	$c_{13}c_{12}$	m''				
$\mathcal{R}$	$(U_1)^3$	R^2	$(RU_1^{-1}RU_1)^3$			
ker			$K(T_2, T_1)$			

$\mathcal{P}6$

$\langle a\rangle$	U_1	U_2	T			
$(\mathrm{hom})^{-1}$	$c_{13}c_{12}$	$c_{13}c_{12}T_1$	S_2			
$\mathcal{R}$	$(U_1)^3$	$(U_2)^3$	$(U_1U_2)^3$	T^2	$TU_1TU_2^{-1}$	
ker			$K(T_2^{-1}, T_1^{-1})$			(16)
$\langle b\rangle$	U_1	T				
$(\mathrm{hom})^{-1}$	$c_{13}c_{12}T_1$	S_2				
$\mathcal{R}$	$(U_1)^3$	T^2	$(U_1T)^6$			
ker			$K(T_2^{-1}, T_1^{-1})$			

$\mathcal{P}6m$

$\langle a\rangle$	R_1	R_2	R_3	R							
$(\mathrm{hom})^{-1}$	$c_{12}T_1$	c_{23}	c_{13}	m''							(17)
$\mathcal{R}$	$(R_1)^2$	$(R_2)^2$	$(R_3)^2$	R^2	$(R_1R_2)^3$	$(R_2R_3)^3$	$(R_3R_1)^3$	RR_1RR_2	RR_3RR_3		
ker					$K(T_1^{-1}, T_2^{-1})$.						

We check the condition (1) given in the introduction. In equation (18) we give for 13 space groups the construction of T_1, T_2 from its generators.

	$\mathcal{P}1$	$\mathcal{P}2$	$\mathcal{P}m$	$\mathcal{C}m$	$\mathcal{P}2mm$	$\mathcal{C}2mm$		
T_1 :	X	.	X	PR	Y^{-1}	TR_2R_3		
T_2 :	Y^{-1}	.	Y^{-1}	$(RP)^{-1}$	$R'R$	R_3TR_2		
	$\mathcal{P}4$	$\mathcal{P}4mm$	$\mathcal{P}3$	$\mathcal{P}3m1$	$\mathcal{P}31m$	$\mathcal{P}6$	$\mathcal{P}6m$	(18)
T_1 :	U_4S^2	R_4R_2	$(U_1)^{-1}U_2$	.	.	$(U_1)^{-1}U_2$	$R_3R_2R_3R_1$	
T_2 :	S^2U_1	R_3R_1	$U_2(U_1)^{-1}$	.	.	$U_2(U_1)^{-1}$	$R_3R_1R_3R_2$.	

It follows that the group $\mathrm{Inn}(F_2)$ forms a subgroup of these 13 groups $\mathcal{G}$. The non-symmorphic group $\mathcal{P}g$ is the first exception. Its generators P, Q are not in $\mathrm{Inn}(F_2)$ but yield the four elements $T_1^2 = PQ$, $T_1T_2 = PQ^{-1}$, $T_2T_1 = P^{-1}Q$, $T_2^2 = P^{-1}Q^{-1}$ of $\mathrm{Inn}(F_2)$. These four elements generate the *subgroup of words of even length* in $\mathrm{Inn}(F_2)$. Due to the presence of the generators P, Q, the same holds true for the groups $\mathcal{P}2mg$, $\mathcal{P}2gg$, $\mathcal{P}4gm$. Upon Abelianization we are of course left with two, not four translations. With this splitting into $13 + 4$, all groups $\mathcal{G}$ match condition (1) given in the introduction.

4. The geometric representation

We pass to a geometric representation of the space groups. It is shown in [5] that the group $\mathrm{Aut}(F_2)$ may be lifted into an isomorphic image in $\mathrm{Aut}(F_3)$. Upon lifting an element of $\mathrm{Aut}(F_2)$ into $\mathrm{Aut}(F_3)$ and then applying hom_2, one obtains a 3×3 block triangular matrix representative for the space group element. In the columns of the following (19) we give for the generators $\langle c_{23}, c_{13}, \sigma_1\rangle$ of the lifted group $\mathrm{Aut}(F_2)$ the images $\langle y_1, y_2, y_3\rangle$ of $\langle x_1, x_2, x_3\rangle$, each followed by the three power sums. The image under hom_2 is then the 3×3 matrix in $GL(3, Z)$ formed from the three rows:

$$\begin{array}{lllllllllllll} e & c_{23} & \mathrm{hom}_2 & & & c_{13} & \mathrm{hom}_2 & & & \sigma_1 & \mathrm{hom}_2 & & \\ x_1 & x_1x_2 & 1 & 1 & 0 & (x_2)^{-1} & 0 & -1 & 0 & (x_1)^{-1} & -1 & 0 & 0 \\ x_2 & (x_2)^{-1} & 0 & -1 & 0 & (x_1)^{-1} & -1 & 0 & 0 & x_2 & 0 & 1 & 0 \\ x_3 & x_2x_3 & 0 & 1 & 1 & x_1x_2x_3 & 1 & 1 & 1 & x_3 & 0 & 0 & 1. \end{array} \tag{19}$$

In the expressions for the space group elements we use the following short-hand symbols, part of which appear in relation to subgroups of $\mathrm{Aut}(F_3)$ [5]. We express these automorphisms as functions of the generators as follows:

$$\begin{array}{lll} c_{12} := c_{23}c_{13}c_{23} & \sigma_2 := c_{13}\sigma_1 c_{13} & m'' := \sigma_1 c_{13}\sigma_1 \\ S_2 := (c_{13}\sigma_1)^2 & S_3 := c_{23}S_2c_{23} & S_1 := c_{12}S_2c_{12} \\ T_1 := S_1S_2 & T_2 := S_2S_3. & \end{array} \tag{20}$$

The action on $\langle x_1, x_2, x_3\rangle$ and the image under hom_2 can be computed by group multiplication from these expressions and from (19). For convenience we give some results of this computation:

$$\begin{array}{lllllllllllll} e & S_2 & \mathrm{hom}_2 & & & T_1 & \mathrm{hom}_2 & & & T_2 & \mathrm{hom}_2 & & \\ x_1 & (x_1)^{-1} & -1 & 0 & 0 & x_1 & 1 & 0 & 0 & (x_2)^{-1}x_1x_2 & 1 & 0 & 0 \\ x_2 & (x_2)^{-1} & 0 & -1 & 0 & (x_1)^{-1}x_2x_1 & 0 & 1 & 0 & x_2 & 0 & 1 & 0 \\ x_3 & x_2^2x_3 & 0 & 2 & 1 & x_1^{-1}x_2^{-1}x_1^{-1}x_2x_3 & -2 & 0 & 1 & x_2^{-2}x_3 & 0 & -2 & 1. \end{array} \tag{21}$$

By applying a matrix from $Gl(3, Z)$ from the left to a column formed by three vectors $\langle x_1, x_2, x_3\rangle$, which geometrically represent the algebraic generators of F_3, one obtains a linear action of the space group element on the plane spanned by $\langle x_1, x_2\rangle$. The third vector can then be suppressed. The automorphisms T_1, T_2 under hom_2 are seen to map into translations by twice the length of the vectors which represent $\langle x_1, x_2\rangle$. In the following figures, the two vectors have been adapted so that the torus or unit cell corresponds to the setting of [9]. The rotation axes, and the generators of mirror and glide lines (full and broken lines) for each of the 17 space groups, are shown on the right, with symbols

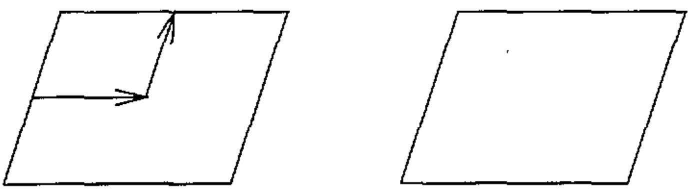

Figure 1. $\mathcal{P}1$ and $P1$.

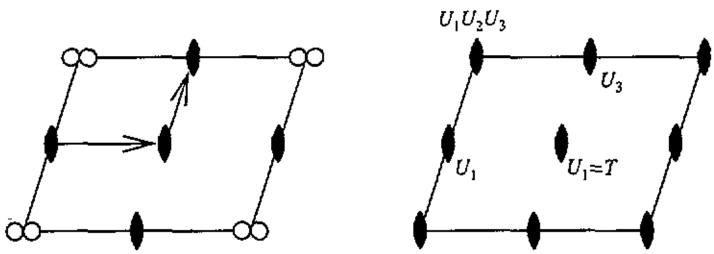

Figure 2. $\mathcal{P}2$ and $P2$.

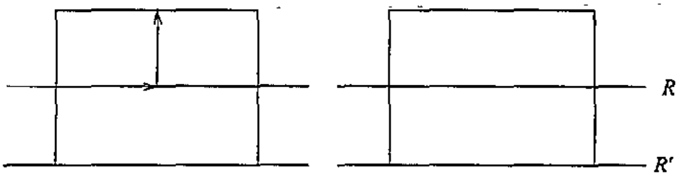

Figure 3. $\mathcal{P}m$ and Pm.

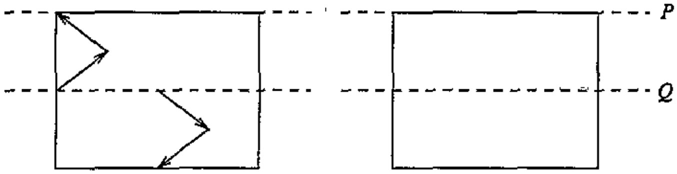

Figure 4. $\mathcal{P}g$ and Pg.

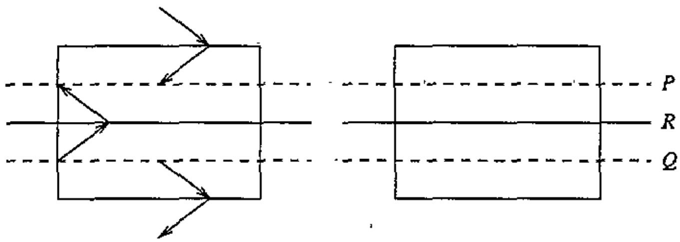

Figure 5. $\mathcal{C}m$ and Cm.

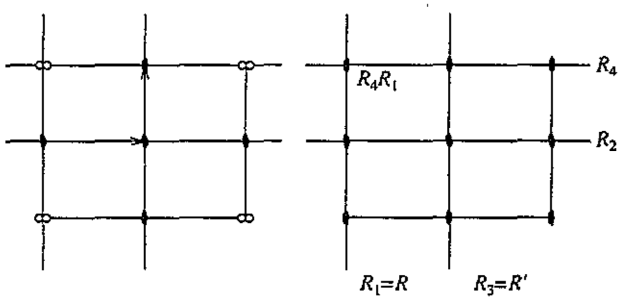

Figure 6. $\mathcal{P}2mm$ and $P2mm$.

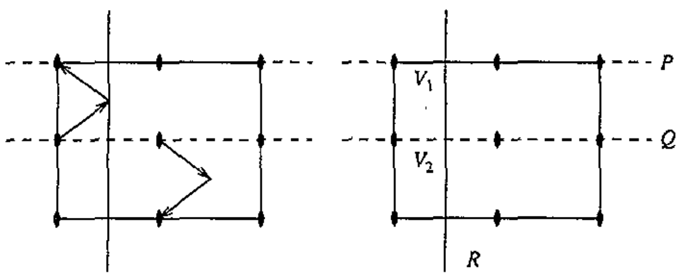

Figure 7. $\mathcal{P}2mg$ and $P2mg$.

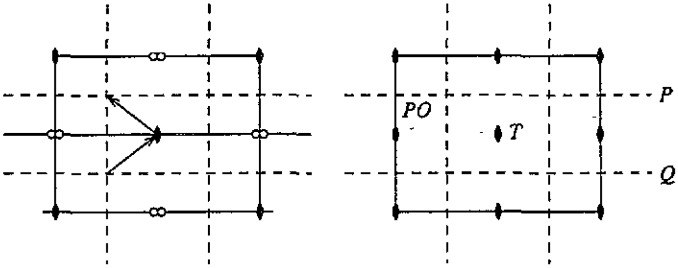

Figure 8. $\mathcal{P}2gg$ and $P2gg$.

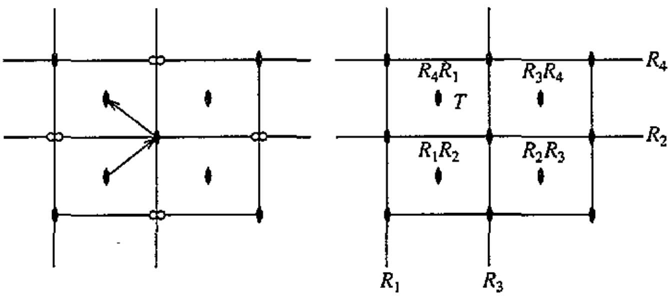

Figure 9. $\mathcal{C}2mm$ and $C2mm$.

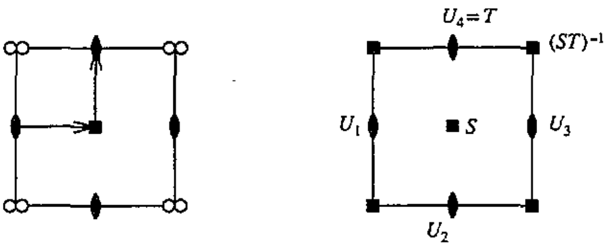

Figure 10. $\mathcal{P}4$ and $P4$.

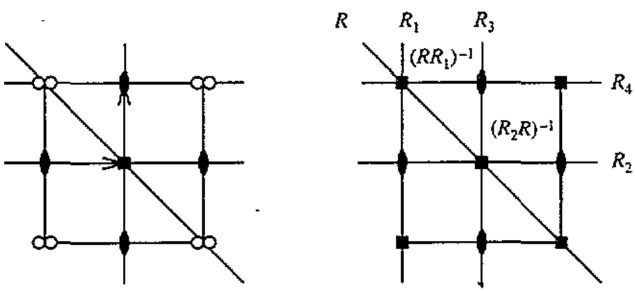

Figure 11. $\mathcal{P}4mm$ and $P4mm$.

which represent typical generators and combinations from [8] and appear in the algebraic expressions of section 3. The left-hand side of each figure gives in the same geometry the two vectors $\langle x_1, x_2 \rangle$, where x_1 starts at the end point of x_2. Symbols ∞ mark the preimages of infinite order in $\mathcal{G}$ for finite-order elements in G.

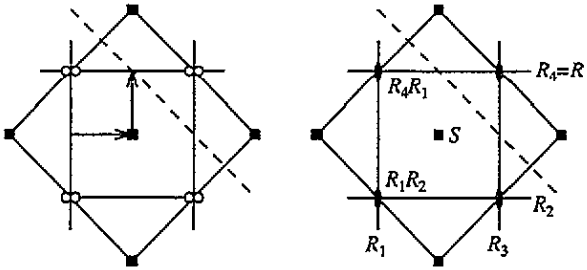

Figure 12. $\mathcal{P}4gm$ and $P4gm$.

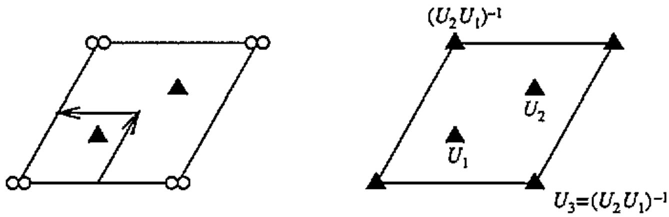

Figure 13. $\mathcal{P}3$ and $P3$.

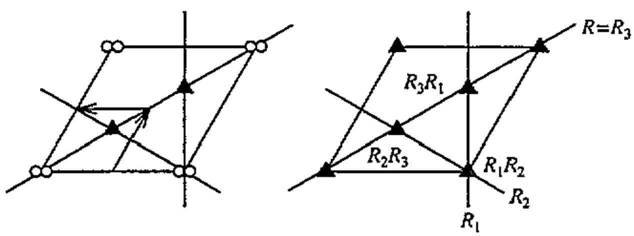

Figure 14. $\mathcal{P}3m1$ and $P3m1$.

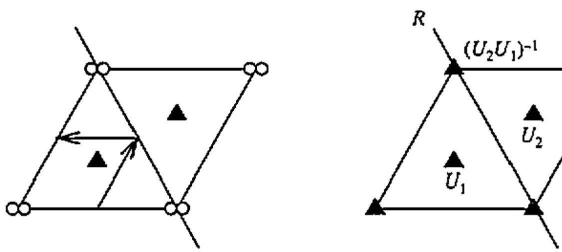

Figure 15. $\mathcal{P}31m$ and $P31m$.

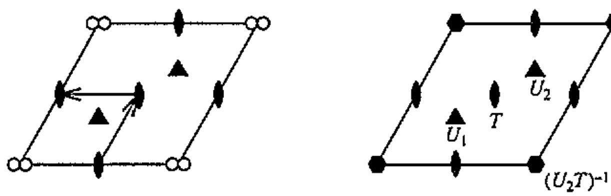

Figure 16. $\mathcal{P}6$ and $P6$.

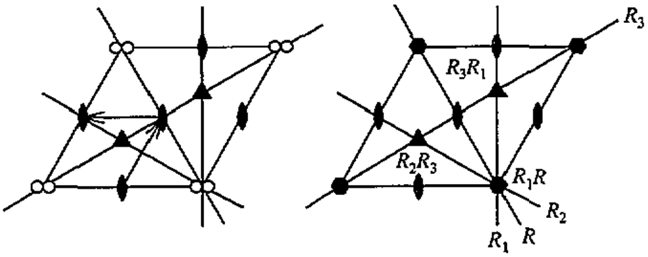

Figure 17. $\mathcal{P}6m$ and $P6m$.

5. Conclusion

We give some comments on the results. The distribution of infinite-order preimages for the rotation axes can be seen from the figures. We omit a corresponding discussion of mirror and glide lines. The group $\mathcal{P}2$ is the universal Coxeter group with three generators. In the groups $\mathcal{P}4gm$, $P4gm$ we mark one representative glide line. In $\mathcal{P}3m1$ compared to the Coxeter group $P3m1$, the product R_1R_2 gets infinite order. There is no element of order 6 in $\mathrm{Aut}(F_2)$ and so the corresponding preimages in $\mathcal{P}6, \mathcal{P}6m$ have order ∞.

If the transversals under the action of the 17 space groups on the plane are connected at corresponding edges, one obtains the 17 *orbifolds* described by Montesinos [3], p 80–2. For $P1$ we again find a torus, and any other orbifold admits an unfolding into an appropriate torus. The orbifold for the example Pg is Klein's bottle. If we extend the notion of the *quantized torus* mentioned in the introduction, each of the 17 groups $\mathcal{G}$ provides a quantum version of the corresponding orbifold.

One field of application for non-commutative crystallography are quasicrystals [6]. Other applications, among them to the theory of dislocations, are under study.

Acknowledgments

It is a pleasure to dedicate the present work to Wolfram Prandl on the occasion of his 60th birthday. The author would like to thank the Physics Department of the Ben-Gurion University of the Negev at Beer Sheva for a Dozor Fellowship in March 1995. The work was completed during this stay. Many thanks are due to Shelomo I Ben-Abraham for helpful suggestions and discussions, to Ingo Brüggemann for checking part of the algebra, and to Tobias Kramer for substantial help in the preparation of the figures.

References

[1] Connes A 1994 *Non-Commutative Geometry* (New York: Academic)
[2] Effros E G 1989 *Math. Intelligencer* **11** 27–34
[3] Montesinos J M 1985 *Classical Tesselations and Three-Manifolds* (Berlin: Springer)
[4] Nielsen J 1924 *Math. Ann.* **91** 169–209
[5] Kramer P 1995 *J. Phys. A: Math. Gen.* **28** 379–89
[6] Kramer P and Garcia-Escudero J 1995 Non-commutative models for quasicrystals *Beyond Quasicrystals* ed F Axel and D Gratias (Berlin: Springer) pp 55–73
[7] McCool J 1980 *Trans. Am. Math. Soc.* **260** 309–18
[8] Coxeter H S M and Moser W O 1965 *Generators and Relations for Discrete Groups* (New York: Springer)
[9] Hahn Th (ed) 1983 *International Tables for Crystallography* vol A (Dordrecht: Reidel) pp 81–99